Difference equation

Solve the following difference equation, including the homogeneous and particular parts. Analyze its stability:

$$y_{n+2} + y_{n+1} - 6y_n = 4^n + n + 1,$$

Solution

The associated homogeneous equation is:

$$y_{n+2} + y_{n+1} - 6y_n = 0.$$

We look for solutions of the form $y_n = r^n$:

$$r^{n+2} + r^{n+1} - 6r^n = 0 \implies r^2 + r - 6 = 0.$$

Solving the characteristic equation:

$$r^2 + r - 6 = 0 \implies r = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2}.$$

We find the roots:

- $r_1 = \frac{-1+5}{2} = 2,$
- $r_2 = \frac{-1-5}{2} = -3.$

The solution to the homogeneous equation is:

$$y_n^{(h)} = A \cdot (2)^n + B \cdot (-3)^n,$$

where A and B are constants.
The non-homogeneous equation is:

$$y_{n+2} + y_{n+1} - 6y_n = 4^n + n + 1.$$

We seek a particular solution that is the sum of particular solutions for each term:

$$y_n^{(p)} = y_n^{(p1)} + y_n^{(p2)}.$$

Part 1: Find $y_n^{(p1)}$ for 4^n

We propose:

$$y_n^{(p1)} = C \cdot 4^n.$$

Substituting into the equation:

$$C \cdot 4^{n+2} + C \cdot 4^{n+1} - 6C \cdot 4^n = 4^n.$$

Simplifying:

$$C \cdot 4^n(16 + 4 - 6) = 4^n \implies C \cdot 14 \cdot 4^n = 4^n.$$

Dividing through by 4^n:

$$14C = 1 \implies C = \frac{1}{14}.$$

Thus:

$$y_n^{(p1)} = \frac{1}{14} \cdot 4^n.$$

Part 2: Find $y_n^{(p2)}$ for $n+1$

We propose a solution of the form:

$$y_n^{(p2)} = Pn + Q.$$

Calculating:

$$\begin{aligned} y_{n+2}^{(p2)} &= P(n+2) + Q = Pn + 2P + Q, \\ y_{n+1}^{(p2)} &= P(n+1) + Q = Pn + P + Q, \\ y_n^{(p2)} &= Pn + Q. \end{aligned}$$

Substituting into the equation:

$$[Pn + 2P + Q] + [Pn + P + Q] - 6[Pn + Q] = n + 1.$$

Simplifying:

$$(2Pn - 6Pn) + (2P + P + Q + Q - 6Q) = n + 1 \implies (-4Pn) + (3P - 4Q) = n + 1.$$

Matching coefficients:

1. Coefficient of n:
$$-4P = 1 \implies P = -\frac{1}{4}.$$

2. Constant term:
$$3P - 4Q = 1 \implies 3\left(-\frac{1}{4}\right) - 4Q = 1 \implies -\frac{3}{4} - 4Q = 1.$$

Solving for Q:

$$-4Q = 1 + \frac{3}{4} \implies -4Q = \frac{7}{4} \implies Q = -\frac{7}{16}.$$

Thus:

$$y_n^{(p2)} = -\frac{1}{4}n - \frac{7}{16}.$$

Total Particular Solution:

$$y_n^{(p)} = y_n^{(p1)} + y_n^{(p2)} = \frac{1}{14} \cdot 4^n - \frac{1}{4}n - \frac{7}{16}.$$

General Solution to the Non-Homogeneous Equation

Adding the homogeneous and particular solutions:

$$y_n = y_n^{(h)} + y_n^{(p)} = A \cdot (2)^n + B \cdot (-3)^n + \frac{1}{14} \cdot 4^n - \frac{1}{4}n - \frac{7}{16}.$$

Stability Analysis

The stability of the solution depends on the behavior of y_n as $n \to \infty$.

- **Terms with $(2)^n$ and $(-3)^n$:** The absolute values of the roots $r = 2$ and $r = -3$ are greater than 1, so these terms grow exponentially.
- **Term with 4^n:** Also grows exponentially.
- **Linear term $-\frac{1}{4}n$:** Grows linearly.
- **Constant term $-\frac{7}{16}$:** Remains constant.

The solution is unstable, as y_n grows without bound as $n \to \infty$.